Mathematical Derivations of Semianalytical Solutions for Hydraulic Tests in Highly Permeable Aquifers

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I. Introduction

In this report, derivations of the transform-space solutions to the mathematical model describing the head response to a hydraulic test in a highly permeable aquifer are presented. For the sake of generality, the solutions are developed in a dimensionless form. The back transforms of these expressions are used by *Butler and Zhan* [in review] to develop new insights into the analysis of hydraulic tests in aquifers of high hydraulic conductivity.

In Section II of this report, governing equations and auxiliary conditions for the aquifer and the test well are presented. Dimensionless variables are introduced in Section III and are used to rewrite the governing equations and auxilary conditions in a dimensionless format. The corresponding dimensionless equations in Laplace-Fourier space are derived in Section IV using standard integral transform methods, and Laplace-space solutions are presented in Section V. In Sections VI-IX, a parallel development is presented for an observation well. The method used to numerically invert the Laplace-space solutions is described in Section X.

II. Governing Equations for Test Well

The problem of interest here is that of the head response produced by a slug or pumping test in a confined aquifer of infinite areal extent and constant thickness. Responses at both the test and observation wells are considered, and the wells may be screened/open across all or a portion of the aquifer. Flow properties are assumed uniform, but the vertical (K_z) and radial (K_r) components of hydraulic conductivity may differ. The inertia of the water column in a well is considered, but inertial effects in the aquifer are assumed negligible (*Bredehoeft et al.* [1966]).

The following derivation will borrow elements from previous work. Inertial mechanisms at the test well will be incorporated following *Kipp* [1985], while inertial mechanisms at the observation well will be incorporated using the approach of *Shapiro* [1989]. The partially penetrating well representations of *Dougherty and Babu* [1984] and *Hyder et al.* [1994] will be used at both the test and observation wells, and frictional losses in the wellbore will be incorporated following *van der Kamp* [1976] and *Ross* [1985]. In all cases, the equations will be written in a general form that is applicable for both pumping and slug tests.

Following the approach of *Kipp* [1985], governing equations and auxiliary conditions can be defined for the test well and aquifer of Figure 1.

Aquifer Flow

$$\frac{\partial^2 h(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial h(r,z,t)}{\partial r} + \frac{K_z}{K_r}\frac{\partial^2 h(r,z,t)}{\partial z^2} = \frac{S_s}{K_r}\frac{\partial h(r,z,t)}{\partial t} \tag{1}$$

$$h(r,z,t=0) = 0 \tag{2}$$

$$h(r=r_w,z,t) = h_s(z,t), \qquad d < z < d+b \tag{3}$$

$$h(r=\infty,z,t) = 0 \tag{4}$$

$$\left.\frac{\partial h(r,z,t)}{\partial z}\right|_{z=0} = 0 \tag{5}$$

$$\left.\frac{\partial h(r,z,t)}{\partial z}\right|_{z=B} = 0 \tag{6}$$

Mass Balance in Test Well

$$\left(\pi\, r_c^2 \frac{dH(t)}{dt} + Q\right)\left(H_v(z-d) - H_v(z-d-b)\right) = 2\pi\, r_w b K_r \left.\frac{\partial h(r,z,t)}{\partial r}\right|_{r=r_w} \tag{7}$$

Momentum Balance in Test Well

$$\frac{d^2 H(t)}{dt^2} + \frac{8\nu L}{r_c^2 L_e}\frac{dH(t)}{dt} + \frac{g}{L_e}H(t) = \frac{g}{L_e b}\int_d^{d+b} h_s(z,t)dz \tag{8}$$

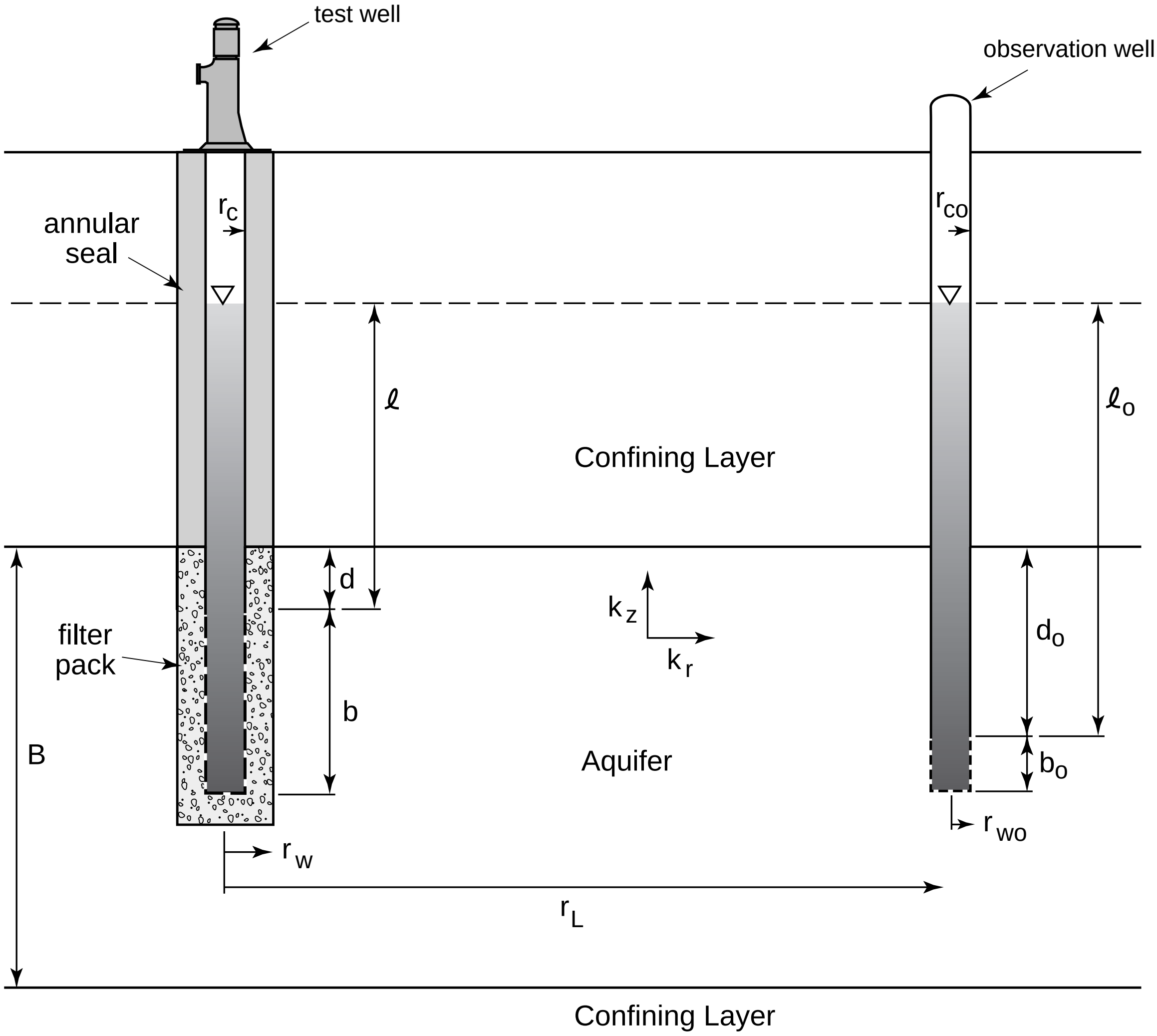
Initial Conditions in Test Well

$$H(t=0) = H_0 \tag{9}$$

$$\left.\frac{dH(t)}{dt}\right|_{t=0} = H_0^{'} - \frac{Q}{\pi\, r_c^2} \tag{10}$$

where
B = aquifer thickness, $[L]$;
b = screen length of test well, $[L]$;
d = distance from top of aquifer to top of screen at test well, $[L]$;
g = gravitational acceleration, $[L/T^2]$;
$H(t)$ = deviation of water level in test well from static conditions, $[L]$;
H_0 = initial deviation of water level in test well from static conditions
(= 0 for pumping test), $[L]$;
$H_0^{'}$ = initial velocity of water level in test well as a result of slug-test initiation, $[L/T]$;
$H_v(z-d)$ = Heaviside function (= 0 for $z-d<0$, = 1 for $z-d>0$);
$h(r,z,t)$ = deviation of hydraulic head in aquifer from static conditions, $[L]$;
$h_s(z,t)$ = deviation of hydraulic head in well screen from static conditions, $[L]$;
$L = l + \frac{b}{2}(\frac{r_c}{r_w})^4$ (*Butler* [2002]), $[L]$;
L_e = effective length of water column in well (*Kipp* [1985]; *Zurbuchen et al.* [2002]), $[L]$;
l = length of water column above the top of the screen, $[L]$;
Q = pumping rate (= 0 for slug test), $[L^3/T]$;
r = radial distance from center of test well, [L];
r_c = radius of well casing for test well, $[L]$;
r_w = radius of well screen for test well, $[L]$;
S_s = specific storage of aquifer, $[L^{-1}]$;
t = time, $[T]$;
ν = kinematic viscosity of water, $[L^2/T]$;
z = vertical distance from aquifer top, increases downward, [L].

In the next section, the dimensionless forms of the above equations will be presented.



not to scale

Figure 1: Cross-sectional view of a hypothetical confined aquifer with test and observation wells.

III. Dimensionless Equations for Test Well

For the purposes of this development, the following dimensionless quantities are defined:

$$\Phi(\tau) = \frac{H(t)}{Q_0} \tag{11}$$

$$\phi_s(\eta, \tau) = \frac{h_s(z, t)}{Q_0} \tag{12}$$

$$\phi(\xi, \eta, \tau) = \frac{h(r, z, t)}{Q_0} \tag{13}$$

$$\Phi_0 = \frac{H_0}{Q_0} \tag{14}$$

$$\Phi_0^{'} = \frac{H_0^{'}}{Q_0} \frac{r_w^2 S_s}{K_r} \tag{15}$$

$$\xi = \frac{r}{r_w} \tag{16}$$

$$\eta = \frac{z}{b} \tag{17}$$

$$\mathcal{B} = \frac{B}{b} \tag{18}$$

$$\zeta = \frac{d}{b} \tag{19}$$

$$\tau = \frac{tK_r}{r_w^2 S_s} \tag{20}$$

$$F_l = \frac{16 b \nu L K_r}{g r_c^4} \tag{21}$$

$$\Psi = \sqrt{\frac{r_w^2 K_z}{b^2 K_r}} \tag{22}$$

$$\alpha = \frac{r_c^2}{2 r_w^2 b S_s} \tag{23}$$

$$\beta = \frac{4 L_e b^2 K_r^2}{g r_c^4} \tag{24}$$

$$q = \frac{Q}{2\pi K_r b Q_0} \tag{25}$$

$$Q_0 = \begin{cases} \frac{Q}{2\pi K_r b} & for\ pumping\ tests, \\ H_0 & for\ slug\ tests. \end{cases} \tag{26}$$

Given the above definitions, the equations for the test well can be written in a dimensionless format as follows:

Aquifer Flow

$$\frac{\partial^2\phi(\xi,\eta,\tau)}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\phi(\xi,\eta,\tau)}{\partial\xi} + \Psi^2\frac{\partial^2\phi(\xi,\eta,\tau)}{\partial\eta^2} = \frac{\partial\phi(\xi,\eta,\tau)}{\partial\tau} \tag{27}$$

$$\phi(\xi,\eta,\tau=0) = 0 \tag{28}$$

$$\phi(\xi=1,\eta,\tau) = \phi_s(\eta,\tau), \quad \zeta < \eta < \zeta+1 \tag{29}$$

$$\phi(\xi=\infty,\eta,\tau) = 0 \tag{30}$$

$$\left.\frac{\partial\phi(\xi,\eta,\tau)}{\partial\eta}\right|_{\eta=0} = 0 \tag{31}$$

$$\left.\frac{\partial\phi(\xi,\eta,\tau)}{\partial\eta}\right|_{\eta=\mathcal{B}} = 0 \tag{32}$$

Mass Balance in Test Well

$$\left(\alpha\frac{d\Phi(\tau)}{d\tau} + q\right)\left(H_v(\eta-\zeta) - H_v(\eta-\zeta-1)\right) = \left.\frac{\partial\phi(\xi,\eta,\tau)}{\partial\xi}\right|_{\xi=1} \tag{33}$$

Momentum Balance in Test Well

$$\alpha^2\beta\frac{d^2\Phi(\tau)}{d\tau^2} + \alpha F_l\frac{d\Phi(\tau)}{d\tau} + \Phi(\tau) = \int_\zeta^{1+\zeta}\phi_s(\eta,\tau)d\eta \tag{34}$$

Initial Conditions in Test Well

$$\Phi(\tau=0) = \Phi_0 \tag{35}$$

$$\left.\frac{d\Phi(\tau)}{d\tau}\right|_{\tau=0} = \Phi_0^{'} - \frac{q}{\alpha} \tag{36}$$

The transform-space forms of the above dimensionless equations are given in the next section.

IV. Transform Space Equations for Test Well

Aquifer Flow

Applying Laplace and finite Fourier cosine transforms to the dimensionless aquifer flow equation yields:

$$\frac{\partial^2 \tilde{\overline{\phi}}(\xi,\omega,p)}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \tilde{\overline{\phi}}(\xi,\omega,p)}{\partial \xi} - \omega^2 \Psi^2 \tilde{\overline{\phi}}(\xi,\omega,p) = p\tilde{\overline{\phi}}(\xi,\omega,p) \tag{37}$$

where $\tilde{\overline{\phi}}(\xi,\omega,p)$ is the Laplace-Fourier transform of $\phi(\xi,\eta,\tau)$; $\omega, p =$ Fourier and Laplace transform variables, respectively.

Mass Balance in Test Well

Applying Laplace and finite Fourier cosine transforms to the dimensionless mass balance equation for the test well yields:

$$\left(-\alpha\Phi_0 + \alpha p \overline{\Phi}(p) + \frac{q}{p}\right) F_c\left(H_v(\eta-\zeta) - H_v(\eta-\zeta-1)\right) = \left.\frac{\partial \tilde{\overline{\phi}}(\xi,\omega,p)}{\partial \xi}\right|_{\xi=1} \tag{38}$$

where $\overline{\Phi}(p)$ is the Laplace transform of $\Phi(\tau)$ and F_c designates a finite Fourier cosine transform.

The finite Fourier cosine transform of the Heaviside function can be written as:

$$\begin{aligned}
& F_c\left(H_v(\eta-\zeta) - H_v(\eta-\zeta-1)\right) \\
= & \int_0^{\mathcal{B}} \left(H_v(\eta-\zeta) - H_v(\eta-\zeta-1)\right)\cos\left(\frac{n\pi\eta}{\mathcal{B}}\right) d\eta \\
= & \int_\zeta^{\zeta+1} \cos\left(\frac{n\pi\eta}{\mathcal{B}}\right) d\eta \\
= & \frac{2\mathcal{B}}{n\pi}\sin\left(\frac{n\pi}{2\mathcal{B}}\right)\cos\left(\frac{n\pi}{2\mathcal{B}} + \frac{n\pi\zeta}{\mathcal{B}}\right) \\
= & \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2} + \omega\zeta\right)
\end{aligned} \tag{39}$$

where

$$\omega = \frac{n\pi}{\mathcal{B}} \tag{40}$$

Momentum Balance in Test Well

Applying the Laplace transform to the dimensionless momentum balance equation for the test well yields:

$$
\begin{aligned}
& \alpha^2\beta\left(-p\Phi_0 - \Phi_0' + \frac{q}{\alpha} + p^2\overline{\Phi}(p)\right) \\
+ \quad & \alpha F_l\left(-\Phi_0 + p\overline{\Phi}(p)\right) + \overline{\Phi}(p) = \int_{\zeta}^{1+\zeta} \overline{\phi_s}(\eta, p)\, d\eta
\end{aligned}
\tag{41}
$$

where $\overline{\phi_s}(\eta, p)$ is the Laplace transform of $\phi_s(\eta, \tau)$.

In the next section, the above equations in transform space will be solved analytically to obtain the Laplace-space functions for the head in the aquifer, the water level in the test well, and the head in the well screen.

V. Laplace Space Solutions for Test Well

Equation (37) is a form of the modified Bessel equation, so a general solution in terms of modified Bessel functions and two constants can be readily found.

$$\tilde{\overline{\phi}}(\xi, \omega, p) = AK_0\left(\sqrt{\Psi^2\omega^2 + p}\xi\right) + BI_0\left(\sqrt{\Psi^2\omega^2 + p}\xi\right) \tag{42}$$

The transform-space form of equation (30) can be used to show that $B = 0$. The other constant, A, is evaluated at $\xi = 1$ using equation (38)

$$A = \frac{\left(-\alpha\Phi_0 + \alpha p\overline{\Phi}(p) + \frac{q}{p}\right) F_c\left(H_v(\eta - \zeta) - H_v(\eta - \zeta - 1)\right)}{K_1\left(\sqrt{\Psi^2\omega^2 + p}\right)\sqrt{\Psi^2\omega^2 + p}} \tag{43}$$

The solution for both $\tilde{\overline{\phi}}(\xi, \omega, p)$ and $\tilde{\overline{\phi_s}}(\omega, p)$ can therefore be written as,

$$\tilde{\overline{\phi}}(\xi, \omega, p) = \left(-\alpha\Phi_0 + \alpha p\overline{\Phi}(p) + \frac{q}{p}\right)$$
$$\frac{F_c\left(H_v(\eta - \zeta) - H_v(\eta - \zeta - 1)\right) K_0\left(\sqrt{\Psi^2\omega^2 + p}\xi\right)}{K_1\left(\sqrt{\Psi^2\omega^2 + p}\right)\sqrt{\Psi^2\omega^2 + p}} \tag{44}$$

$$\tilde{\overline{\phi_s}}(\omega, p) = \left(-\alpha\Phi_0 + \alpha p\overline{\Phi}(p) + \frac{q}{p}\right)$$
$$\frac{F_c\left(H_v(\eta - \zeta) - H_v(\eta - \zeta - 1)\right) K_0\left(\sqrt{\Psi^2\omega^2 + p}\right)}{K_1\left(\sqrt{\Psi^2\omega^2 + p}\right)\sqrt{\Psi^2\omega^2 + p}} \tag{45}$$

After performing an inverse finite Fourier cosine transform, the above equations become

$$\overline{\phi}(\xi, \eta, p) = \left(-\alpha\Phi_0 + \alpha p\overline{\Phi}(p) + \frac{q}{p}\right)\Omega_a\left(\xi, \eta, p\right) \tag{46}$$

$$\overline{\phi_s}(\eta, p) = \left(-\alpha\Phi_0 + \alpha p\overline{\Phi}(p) + \frac{q}{p}\right)\Omega_a\left(\xi = 1, \eta, p\right) \tag{47}$$

where

$$
\begin{aligned}
&\Omega_a(\xi,\eta,p)\\
=\;& F_c^{-1}\left[\frac{F_c\left(H_v(\eta-\zeta)-H_v(\eta-\zeta-1)\right)K_0\left(\sqrt{\Psi^2\omega^2+p}\,\xi\right)}{K_1\left(\sqrt{\Psi^2\omega^2+p}\right)\sqrt{\Psi^2\omega^2+p}}\right]\\
=\;& F_c^{-1}\left[\frac{\frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}+\omega\zeta\right)K_0\left(\sqrt{\Psi^2\omega^2+p}\,\xi\right)}{K_1\left(\sqrt{\Psi^2\omega^2+p}\right)\sqrt{\Psi^2\omega^2+p}}\right]\\
=\;& F_c^{-1}\left(\frac{\frac{2\mathcal{B}}{n\pi}\sin\left(\frac{n\pi}{2\mathcal{B}}\right)\cos\left(\frac{n\pi}{2\mathcal{B}}+\frac{n\pi\zeta}{\mathcal{B}}\right)K_0\left(\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}\,\xi\right)}{K_1\left(\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}\right)\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}}\right)\\
=\;& F_c^{-1}\left(f_c(\xi,\eta,p,n)\right)\\
=\;& \frac{1}{\mathcal{B}}f_c(\xi,\eta,p,n=0)+\frac{2}{\mathcal{B}}\sum_{n=1}^{\infty}f_c(\xi,\eta,p,n)\cos\left(\frac{n\pi}{\mathcal{B}}\eta\right)\\
=\;& \frac{1}{\mathcal{B}}\frac{K_0\left(\sqrt{p}\,\xi\right)}{K_1\left(\sqrt{p}\right)\sqrt{p}}\\
&+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2\mathcal{B}}\right)\cos\left(\frac{n\pi}{2\mathcal{B}}+\frac{n\pi\zeta}{\mathcal{B}}\right)K_0\left(\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}\,\xi\right)}{nK_1\left(\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}\right)\sqrt{\Psi^2\left(\frac{n\pi}{\mathcal{B}}\right)^2+p}}\cos\left(\frac{n\pi}{\mathcal{B}}\eta\right)
\end{aligned}
\tag{48}
$$

Equation (47) is substituted into equation (41) to eliminate $\overline{\phi_s}(\eta,p)$ and solve for $\overline{\Phi}(p)$. The solution for $\overline{\Phi}(p)$ can be written as,

$$
\overline{\Phi}(p)=\frac{-\alpha\beta pq+\alpha pF_l\Phi_0+\alpha^2\beta p^2\Phi_0+\alpha^2\beta p\Phi_0^{'}-(q-\alpha p\Phi_0)\Omega_w(p)}{p\,\left(1+\alpha pF_l+\alpha^2\beta p^2+\alpha p\Omega_w(p)\right)}
\tag{49}
$$

where

$$
\Omega_w(p)=\int_{\zeta}^{\zeta+1}\Omega_a(\xi=1,\eta,p)\,d\eta
\tag{50}
$$

Substituting the solution for $\overline{\Phi}(p)$ into equation (46) and (47) leads to the following solution for $\overline{\phi}(\xi,\eta,p)$ and $\overline{\phi_s}(\eta,p)$, respectively:

$$
\overline{\phi}(\xi,\eta,p)=\frac{\left(\alpha p\Phi_0-\alpha^3\beta p^2\Phi_0^{'}-q-\alpha pF_lq\right)\,\Omega_a(\xi,\eta,p)}{p\,\left(1+\alpha pF_l+\alpha^2\beta p^2+\alpha p\Omega_w(p)\right)}
\tag{51}
$$

$$\overline{\phi_s}(\eta, p) = \frac{\left(\alpha p \Phi_0 - \alpha^3 \beta p^2 \Phi_0^{'} - q - \alpha p F_l q\right) \Omega_a \left(\xi = 1, \eta, p\right)}{p \, \left(1 + \alpha p F_l + \alpha^2 \beta p^2 + \alpha p \Omega_w \left(p\right)\right)} \tag{52}$$

The above three solutions in Laplace space are difficult to invert analytically, so a numerical inversion scheme must be used. In this work, the method of *D'Amore et al.* [1999a, b] is employed to numerically invert the Laplace-space solutions. That method is described in Section X.

VI. Governing Equations for Observation Well

The preceding development assumed that inertial effects due to the water column at the observation well are negligible. However, *Shapiro* [1989] demonstrated that inertial mechanisms at the observation well are of practical importance in many situations. Thus, a general model must incorporate inertial mechanisms at both the test and observation wells. Following the approach of *Shapiro* [1989], governing equations and auxiliary conditions can be defined for the observation well and aquifer in Figure 1.

Aquifer Flow

$$\frac{\partial^2 h_o(r_o,z,t)}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial h_o(r_o,z,t)}{\partial r_o} + \frac{K_z}{K_r}\frac{\partial^2 h_o(r_o,z,t)}{\partial z^2} = \frac{S_s}{K_r}\frac{\partial h_o(r_o,z,t)}{\partial t} \tag{53}$$

$$h_o(r_o,z,t=0) = 0 \tag{54}$$

$$h_o(r_o = r_{wo},z,t) = h_{so}(z,t), \qquad d_o < z < d_o + b_o \tag{55}$$

$$h_o(r_o = \infty,z,t) = 0 \tag{56}$$

$$\left.\frac{\partial h_o(r_o,z,t)}{\partial z}\right|_{z=0} = 0 \tag{57}$$

$$\left.\frac{\partial h_o(r_o,z,t)}{\partial z}\right|_{z=B} = 0 \tag{58}$$

Mass Balance in Observation Well

$$\begin{aligned} & \left(\pi\, r_{co}^2 \frac{dW_o(t)}{dt}\right)\left(H_v(z-d_o) - H_v(z-d_o-b_o)\right) \\ = \;& 2\pi\, r_{wo} b_o K_r \left.\frac{\partial h_o(r_o,z,t)}{\partial r_o}\right|_{r_o=r_{wo}} \end{aligned} \tag{59}$$

Momentum Balance in Observation Well

$$\begin{aligned} & \frac{d^2 W_o(t)}{dt^2} + \frac{8\nu\, L_o}{r_{co}^2 L_{eo}}\frac{dW_o(t)}{dt} + \frac{g}{L_{eo}} W_o(t) \\ = \;& \frac{g}{L_{eo} b_o}\int_{d_o}^{d_o+b_o} \left(h\left(r = r_L,z,t\right) + h_{so}(z,t)\right) dz \end{aligned} \tag{60}$$

Initial Conditions at Observation Well

$$W_o(t=0) = 0 \tag{61}$$

$$\left.\frac{dW_o(t)}{dt}\right|_{t=0} = 0 \tag{62}$$

where
r_L = distance from test well to observation well, [L];
r_o = radial distance from center of observation well, [L];
$W_o(t)$ = deviation of water level in observation well from static conditions, [L].

All other parameters with a subscript o are the observation-well equivalents to the parameters defined in Section II.

The next section will present the dimensionless forms of the above equations.

VII. Dimensionless Equations for Observation Well

For the purposes of this development, the following additional dimensionless quantities are defined:

$$\Phi_o(\tau) = \frac{W_o(t)}{Q_0} \tag{63}$$

$$\phi_{so}(\eta_o, \tau) = \frac{h_{so}(z, t)}{Q_0} \tag{64}$$

$$\phi_o(\xi_o, \eta_o, \tau) = \frac{h_o(r_o, z, t)}{Q_0} \tag{65}$$

$$\xi_o = \frac{r_o}{r_{wo}} \tag{66}$$

$$\eta_o = \frac{z}{b_o} \tag{67}$$

$$\mathcal{B}_o = \frac{B}{b_o} \tag{68}$$

$$\zeta_o = \frac{d_o}{b_o} \tag{69}$$

$$F_{lo} = \frac{16 b_o \nu L_o K_r}{g r_{co}^4} \tag{70}$$

$$\Psi_o = \sqrt{\frac{r_{wo}^2 K_z}{b_o^2 K_r}} \tag{71}$$

$$\alpha_o = \frac{r_{co}^2}{2 r_{wo}^2 b_o S_s} \tag{72}$$

$$\beta_o = \frac{4 L_{eo} b_o^2 K_r^2}{g r_{co}^4} \tag{73}$$

$$\mathcal{R}_w = \frac{r_{wo}^2}{r_w^2} \tag{74}$$

$$\xi_L = \frac{r_L}{r_w} \tag{75}$$

$$\gamma_b = \frac{b_o}{b} \tag{76}$$

Given these definitions and those in Section III, the equations for the observation well can be written in a dimensionless format as follows:

Aquifer Flow

$$\frac{\partial^2 \phi_o(\xi_o,\eta_o,\tau)}{\partial \xi_o^2} + \frac{1}{\xi_o}\frac{\partial \phi_o(\xi_o,\eta_o,\tau)}{\partial \xi_o} + \Psi_o^2 \frac{\partial^2 \phi_o(\xi_o,\eta_o,\tau)}{\partial \eta_o^2} = \mathcal{R}_w \frac{\partial \phi_o(\xi_o,\eta_o,\tau)}{\partial \tau} \tag{77}$$

$$\phi_o(\xi_o,\eta_o,\tau=0) = 0 \tag{78}$$

$$\phi_o(\xi_o=1,\eta_o,\tau) = \phi_{so}(\eta_o,\tau), \quad \zeta_o < \eta_o < \zeta_o + 1 \tag{79}$$

$$\phi_o(\xi_o=\infty,\eta_o,\tau) = 0 \tag{80}$$

$$\left.\frac{\partial \phi_o(\xi_o,\eta_o,\tau)}{\partial \eta_o}\right|_{\eta_o=0} = 0 \tag{81}$$

$$\left.\frac{\partial \phi_o(\xi_o,\eta_o,\tau)}{\partial \eta_o}\right|_{\eta_o=\mathcal{B}_o} = 0 \tag{82}$$

Mass Balance in Observation Well

$$\alpha_o \mathcal{R}_w \frac{d\Phi_o(\tau)}{d\tau}\left(H_v(\eta_o-\zeta_o) - H_v(\eta_o-\zeta_o-1)\right) = \left.\frac{\partial \phi_o(\xi_o,\eta_o,\tau)}{\partial \xi_o}\right|_{\xi_o=1} \tag{83}$$

Momentum Balance in Observation Well

$$\begin{aligned} & \alpha_o^2 \beta_o \mathcal{R}_w^2 \frac{d^2\Phi_o(\tau)}{d\tau^2} + \alpha_o \mathcal{R}_w F_{lo} \frac{d\Phi_o(\tau)}{d\tau} + \Phi_o(\tau) \\ = & \int_{\zeta_o}^{1+\zeta_o} \left(\phi(\xi=\xi_L,\ \gamma_b\eta_o,\tau) + \phi_{so}(\eta_o,\tau)\right)\, d\eta_o \end{aligned} \tag{84}$$

Initial Conditions in Observation Well

$$\Phi_o(\tau=0) = 0 \tag{85}$$

$$\left.\frac{d\Phi_o(\tau)}{d\tau}\right|_{\tau=0} = 0 \tag{86}$$

The above dimensionless equations are transformed into Laplace and Fourier space in the next section.

VIII. Transform Space Equations for Observation Well

Aquifer Flow

Applying Laplace and finite Fourier cosine transforms to the dimensionless aquifer flow equation yields:

$$\frac{\partial^2 \widetilde{\overline{\phi_o}}(\xi_o, \omega_o, p)}{\partial \xi_o^2} + \frac{1}{\xi_o} \frac{\partial \widetilde{\overline{\phi_o}}(\xi_o, \omega_o, p)}{\partial \xi_o} - \omega_o^2 \Psi_o^2 \widetilde{\overline{\phi_o}}(\xi_o, \omega_o, p) = \mathcal{R}_w p \widetilde{\overline{\phi_o}}(\xi_o, \omega_o, p) \tag{87}$$

where $\widetilde{\overline{\phi_o}}(\xi_o, \omega_o, p)$ is the Laplace-Fourier transform of $\phi_o(\xi_o, \eta_o, \tau)$; ω_o, p = Fourier and Laplace transform variables, respectively.

Mass Balance in Observation Well

Applying Laplace and finite Fourier cosine transforms to the dimensionless mass balance equation for the observation well yields:

$$\alpha_o \mathcal{R}_w p \overline{\Phi_o}(p) F_c \left(H_v(\eta_o - \zeta_o) - H_v(\eta_o - \zeta_o - 1) \right) = \left. \frac{\partial \widetilde{\overline{\phi_o}}(\xi_o, \eta_o, p)}{\partial \xi_o} \right|_{\xi_o = 1} \tag{88}$$

where $\overline{\Phi_o}(p)$ is the Laplace transform of $\Phi_o(\tau)$ and F_c designates a finite Fourier cosine transform.

The finite Fourier cosine transform of the Heaviside function can be written as:

$$\begin{aligned}
& F_c \left(H_v(\eta_o - \zeta_o) - H_v(\eta_o - \zeta_o - 1) \right) \\
= & \int_0^{\mathcal{B}_o} \left(H_v(\eta_o - \zeta_o) - H_v(\eta_o - \zeta_o - 1) \right) \cos\left(\frac{n\pi\eta_o}{\mathcal{B}_o} \right) d\eta_o \\
= & \int_{\zeta_o}^{\zeta_o + 1} \cos\left(\frac{n\pi\eta_o}{\mathcal{B}_o} \right) d\eta_o \\
= & \frac{2\mathcal{B}_o}{n\pi} \sin\left(\frac{n\pi}{2\mathcal{B}_o} \right) \cos\left(\frac{n\pi}{2\mathcal{B}_o} + \frac{n\pi\zeta_o}{\mathcal{B}_o} \right) \\
= & \frac{2}{\omega_o} \sin\left(\frac{\omega_o}{2} \right) \cos\left(\frac{\omega_o}{2} + \omega_o \zeta_o \right)
\end{aligned} \tag{89}$$

where

$$\omega_o = \frac{n\pi}{\mathcal{B}_o} \tag{90}$$

Momentum Balance in Observation Well

Applying the Laplace transform to the dimensionless momentum balance equation for the observation well yields:

$$
\begin{aligned}
& \left(\alpha_o^2 \beta_o \mathcal{R}_w^2 p^2 + \alpha_o \mathcal{R}_w F_{lo} p + 1\right) \overline{\Phi_o}(p) \\
= & \int_{\zeta_o}^{1+\zeta_o} \left(\overline{\phi}(\xi = \xi_L,\, \gamma_b \eta_o, p) + \overline{\phi}_{so}(\eta_o, p)\right) \, d\eta_o
\end{aligned} \tag{91}
$$

where $\overline{\phi_{so}}(\eta_o, p)$ is the Laplace transform of $\phi_{so}(\eta_o, \tau)$.

In the next section, the above equations in transform space will be solved analytically to obtain the Laplace-space functions for the head in the aquifer, the water level in the observation well, and the head in the observation well screen.

IX. Laplace Space Solutions for Observation Well

The Laplace space solutions for the observation well can be obtained following the same steps as outlined in Section V and can be written as follows:

The head in the aquifer:

$$\overline{\phi_o}(\xi_o,\eta_o,p) = -\frac{\alpha_o\mathcal{R}_w p\Omega_{aL}(p)\Omega_o\left(\xi_o,\eta_o,p\right)}{1+\alpha_o\mathcal{R}_w pF_{lo}+\alpha_o^2\beta_o\mathcal{R}_w^2p^2+\alpha_o\mathcal{R}_w p\Omega_{ow}(p)} \tag{92}$$

where

$$\begin{aligned}
&\Omega_o\left(\xi_o,\eta_o,p\right)\\
=\;& F_c^{-1}\left[\frac{F_c\left(H_v(\eta_o-\zeta_o)-H_v(\eta_o-\zeta_o-1)\right)K_0\left(\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}\,\xi_o\right)}{K_1\left(\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}\right)\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}}\right]\\
=\;& F_c^{-1}\left[\frac{\frac{2}{\omega_o}\sin\left(\frac{\omega_o}{2}\right)\cos\left(\frac{\omega_o}{2}+\omega_o\zeta_o\right)K_0\left(\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}\,\xi_o\right)}{K_1\left(\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}\right)\sqrt{\Psi_o^2\omega_o^2+\mathcal{R}_w p}}\right]\\
=\;& F_c^{-1}\left[\frac{\frac{2\mathcal{B}_o}{n\pi}\sin\left(\frac{n\pi}{2\mathcal{B}_o}\right)\cos\left(\frac{n\pi}{2\mathcal{B}_o}+\frac{n\pi\zeta_o}{\mathcal{B}_o}\right)K_0\left(\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}\,\xi_o\right)}{K_1\left(\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}\right)\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}}\right]\\
=\;& F_c^{-1}\left(f_c\left(\xi_o,\eta_o,p,n\right)\right)\\
=\;& \frac{1}{\mathcal{B}_o}f_c\left(\xi_o,\eta_o,p,n=0\right)+\frac{2}{\mathcal{B}_o}\sum_{n=1}^{\infty}f_c\left(\xi_o,\eta_o,p,n\right)\cos\left(\frac{n\pi}{\mathcal{B}_o}\eta_o\right)\\
=\;& \frac{1}{\mathcal{B}_o}\frac{K_0\left(\sqrt{\mathcal{R}_w p}\,\xi_o\right)}{K_1\left(\sqrt{\mathcal{R}_w p}\right)\sqrt{\mathcal{R}_w p}}\\
&+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2\mathcal{B}_o}\right)\cos\left(\frac{n\pi}{2\mathcal{B}_o}+\frac{n\pi\zeta_o}{\mathcal{B}_o}\right)K_0\left(\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}\,\xi_o\right)}{nK_1\left(\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}\right)\sqrt{\Psi_o^2\left(\frac{n\pi}{\mathcal{B}_o}\right)^2+\mathcal{R}_w p}}\cos\left(\frac{n\pi}{\mathcal{B}_o}\eta_o\right)
\end{aligned} \tag{93}$$

$$\Omega_{ow}\left(p\right) = \int_{\zeta_o}^{\zeta_o+1}\Omega_o\left(\xi_o=1,\eta_o,p\right)d\eta_o \tag{94}$$

$$\Omega_{aL}\left(p\right) = \int_{\zeta_o}^{\zeta_o+1}\overline{\phi}\left(\xi=\xi_L,\gamma_b\eta_o,p\right)d\eta_o \tag{95}$$

The water level in the observation well:

$$\overline{\Phi_o}(p) = \frac{\Omega_{aL}(p)}{1+\alpha_o\mathcal{R}_w pF_{lo}+\alpha_o^2\beta_o\mathcal{R}_w^2p^2+\alpha_o\mathcal{R}_w p\Omega_{ow}(p)} \tag{96}$$

The head in the observation well screen:

$$\overline{\phi_{so}}(\eta_o, p) = -\frac{\alpha_o \mathcal{R}_w p \Omega_{aL}(p) \Omega_o\left(\xi_o = 1, \eta_o, p\right)}{1 + \alpha_o \mathcal{R}_w p F_{lo} + \alpha_o^2 \beta_o \mathcal{R}_w^2 p^2 + \alpha_o \mathcal{R}_w p \Omega_{ow}(p)} \tag{97}$$

The above three solutions in Laplace space are difficult to invert analytically, so a numerical inversion scheme must be used. In this work, the method of *D'Amore et al.* [1999a, b], described in the next section, is used to numerically invert the Laplace-space solutions.

X. Numerical Inversion of Laplace Space Solutions

The solutions in Laplace space given in the previous sections are most readily evaluated using a numerical inversion scheme. The *Stehfest* [1970] algorithm, the most commonly used inversion algorithm for well hydraulics applications, cannot invert oscillatory functions accurately (*Kipp* [1985]), so other approaches must be used for the inversion of head responses in highly permeable aquifers. *Kipp* [1985] and *Shapiro* [1989] used the *Crump* [1976] algorithm to successfully perform the inversion of fully penetrating well solutions that incorporated inertial mechanisms at a single well. However, Kipp reported that the Crump method occasionally terminated prior to convergence. *D'Amore et al.* [1999a, b] showed that the Crump method depends heavily on the choice of the computational parameters, and demonstrated that the Crump method does not always converge. As an alternative, they presented a Fourier series method for the numerical inversion of a Laplace-space function. This approach, which is based on the *de Hoog et al.* [1982] improvement of the *Crump* [1976] method, facilitates the determination of the parameters on which numerical performance, accuracy and efficiency depend. In addition, step size and integration boundaries are adjusted automatically to satisfy the desired tolerances. Convergence is guaranteed and is obtained at a near minimum computation cost. Given these advantages over the Crump method, the D'Amore *et al.* method was selected as the numerical inversion approach for this work.

The D'Amore *et al.* method obtains a real–valued function $f(t)$ from the complex Laplace-space function $F(z)$. The relation between $F(z)$ and $f(t)$ is given by the Laplace transformation.

$$F\left(z\right)=\int_0^\infty e^{-zt}f(t)dt,\quad z=\sigma+iy,\quad Re\left(z\right)>\sigma_0,\quad t>0 \tag{98}$$

or the Riemann inversion formula

$$f(t)=\frac{1}{2\,\pi\,i}\int_{\sigma-i\infty}^{\sigma+i\infty}F\left(z\right)e^{zt}dz,\quad z=\sigma\,+\,i\,y,Re\left(z\right)\,>\,\sigma_0,t>0 \tag{99}$$

For computation purposes, equation (99) can be written as

$$f(t)=\frac{e^{\sigma\,t}}{\pi}\int_0^\infty Re\left(F\left(z\right)\,e^{i\,y\,t}\right)dy \tag{100}$$

This equation is obtained through the following steps:

$$\begin{aligned}
F\left(z\right) &= Re\left(F\left(z\right)\right)+iIm\left(F\left(z\right)\right)\\
&= \int_0^\infty e^{-(\sigma+iy)\,t}f(t)dt\\
&= \int_0^\infty e^{-\sigma\,t}e^{-\,i\,y\,t}f(t)dt\\
&= \int_0^\infty e^{-\sigma\,t}\,\left(\cos(yt)+\,i\,\sin(yt)\right)f(t)dt\\
&= \int_0^\infty e^{-\sigma\,t}\,\cos(yt)f(t)dt\,+\,i\int_0^\infty e^{-\sigma\,t}\,\sin(yt)f(t)dt
\end{aligned} \tag{101}$$

The real and imaginary portions of the above equation are Fourier cosine and sine transforms, respectively. Performing the inverse Fourier cosine and sine transforms leads to

$$
\begin{aligned}
f(t) &= \frac{2e^{\sigma\, t}}{\pi} \int_0^{\infty} \cos(yt) Re\left(F\left(z\right)\right) dy &\quad (102)\\
f(t) &= -\frac{2e^{\sigma\, t}}{\pi} \int_0^{\infty} \sin(yt) Im\left(F\left(z\right)\right) dy &\quad (103)
\end{aligned}
$$

Combining the above two equations resuls in

$$
\begin{aligned}
f(t) &= \frac{e^{\sigma\, t}}{\pi} \int_0^{\infty} \left(\cos(yt) Re\left(F\left(z\right)\right) - \sin(yt) Im\left(F\left(z\right)\right)\right) dy \\
&= \frac{e^{\sigma\, t}}{\pi} \int_0^{\infty} Re\left(F\left(z\right)\, e^{i\, y\, t}\right) dy \\
&= \frac{e^{\sigma\, t}}{\pi} Re\left(\int_0^{\infty} F\left(z\right)\, e^{i\, y\, t} dy\right) &\quad (104)
\end{aligned}
$$

Using the trapezoidal rule with step size π/T or the Fourier series method with period $2T$ for the integral in the above equation gives the D'Amore *et al.* inverse Laplace transform algorithm.

$$
\begin{aligned}
f(t) &\approx \frac{1}{T} e^{\sigma\, t} Re\left(\frac{F\left(\sigma\right)}{2} + \sum_{k=1}^{\infty} F\left(\sigma\, +\, i\, \frac{k\, \pi}{T}\right)\, e^{i\, \frac{k\, \pi}{T}\, t}\right) \\
&\approx \frac{1}{T} e^{\sigma\, t} Re\left(\frac{F\left(\sigma\right)}{2} + \sum_{k=1}^{N} F\left(\sigma\, +\, i\, \frac{k\, \pi}{T}\right)\, e^{i\, \frac{k\, \pi}{T}\, t}\right) &\quad (105)
\end{aligned}
$$

Readers are referred to *D'Amore et al.* [1999a] for a description of the automatic procedure used to determine the parameters σ, T, and N in the above equation.

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